Indian Statistical Institute, Bangalore

B.Math (Hons.) I Year, First Semester

Backpaper Examination

Time: 3 hours — Dec 2011 Instructor: C.R.E.Raja

Analysis I Maximum marks: 50

Answer all questions, each question is worth 10 marks

1. If (a_n) and (b_n) converge, prove that:

   i)$(a_n \pm b_n)$, (ca_n) and (a_nb_n) converge for any constant c and

   ii)(a_n) is bounded.

2. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Prove that f is continuous at $x \in \mathbb{R}$ if and only if $\lim f(x_n) = f(x)$ for any sequence $x_n \to x$.

   (b) If $f\colon I \to \mathbb{R}$ is continuous and $x, y \in I$ with s lying between $f(x)$ and $f(y)$, prove that there is a t between x and y such that $f(t) = s$.

3. Let $f\colon (a, b) \to \mathbb{R}$ be a uniformly continuous function. Then prove the following:

   (a)$\lim_{x\to a} f(x)$ exists,

   (b) there is a continuous function $F\colon [a, b] \to \mathbb{R}$ such that $F = f$ on (a, b).

   (c) f is bounded.

4. (a) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be functions such that $f'(x), g'(x)$ exist with $g'(x) \neq 0$ and $f(x) = 0 = g(x)$ for some $x \in \mathbb{R}$. Prove that $\lim_{t\to x} f(t)/g(t) = \frac{f'(x)}{g'(x)}$.

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that f is convex if and only if f' is increasing.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function.

   (a) If $f'(t) \neq 1$ for all $t \in \mathbb{R}$, prove that there is at most one $x \in \mathbb{R}$ such that $f(x) = x$.

   (b) If $\sup_{t\in\mathbb{R}} |f'(t)| < 1$, prove that f has a unique $x \in \mathbb{R}$ such that $f(x) = x$.